Finslerian 3-spinors and the generalized Duffin–Kemmer equation

A. V. Solov'yov

Division of Theoretical Physics, Faculty of Physics, Moscow State University, Moscow, Russia

Abstract

The main facts of the geometry of Finslerian 3-spinors are formulated. The close connection between Finslerian 3-spinors and vectors of the 9-dimensional linear Finslerian space is established. The isometry group of this space is described. The procedure of dimensional reduction to 4-dimensional quantities is formulated. The generalized Duffin–Kemmer equation for a Finslerian 3-spinor wave function of a free particle in the momentum representation is obtained. From the viewpoint of a 4-dimensional observer, this 9-dimensional equation splits into the standard Dirac and Klein–Gordon equations.

Introduction

In the works [1, 2], *hyperspinors* and their basic properties were considered. The same mathematical objects were independently studied under the name of N-component spinors in the papers [3, 4]. Finally, in the work [5], the general algebraic theory of *Finslerian N-spinors* was constructed. The last term is more suitable because it reflects the close connection between hyperspinors and Finslerian geometry.

This paper is devoted to formulating the main facts of the geometry of Finslerian 3-spinors and deducing the generalized Duffin–Kemmer equation for a Finslerian 3-spinor wave function of a free particle in the momentum representation.

In short, the paper has the following structure. We begin with the definition of the space of Finslerian 3-spinors and construct the associated Finslerian geometry. After deducing the expression for the length of a vector in the 9-dimensional linear Finslerian space, we describe the corresponding

isometry group. We also formulate the procedure of dimensional reduction which allows us to rewrite the expression for the Finslerian length of a 9-vector in terms of 4-dimensional geometric objects. We end the paper with deducing the generalized Duffin–Kemmer equation for a Finslerian 3-spinor wave function of a free particle in the momentum representation.

The geometry of Finslerian 3-spinors

Let $\mathbb{C}^3$ be the linear space of 3-component columns of complex numbers with respect to the standard matrix operations of addition and multiplication by elements of the field $\mathbb{C}$. Let us consider the antisymmetric 3-linear form

$$[\xi, \eta, \lambda] = \varepsilon_{abc}\, \xi^a \eta^b \lambda^c, \tag{1}$$

where ξ, η, $\lambda \in \mathbb{C}^3$, ε_{abc} is the Levi-Civita symbol with the ordinary normalization $\varepsilon_{123} = 1$, the indices a, b, c run independently from 1 to 3, and ξ^a, η^b, $\lambda^c \in \mathbb{C}$. Here and in the following formulas, the summation is taken over all the repeating indices.

The space $\mathbb{C}^3$ equipped with the form (1) is called the *space of Finslerian 3-spinors*. The complex number $[\xi, \eta, \lambda]$ is respectively called the *symplectic scalar 3-product* of the Finslerian 3-spinors ξ, η, and λ.

Since (1) is the determinant

$$[\xi, \eta, \lambda] = \begin{vmatrix} \xi^1 & \eta^1 & \lambda^1 \\ \xi^2 & \eta^2 & \lambda^2 \\ \xi^3 & \eta^3 & \lambda^3 \end{vmatrix} \tag{2}$$

with the columns ξ, η, and λ, the symplectic scalar 3-product $[\xi, \eta, \lambda]$ vanishes if and only if the Finslerian 3-spinors ξ, η, and λ are linearly dependent [6]. In particular, $[\xi, \xi, \xi] = 0$ for any $\xi \in \mathbb{C}^3$.

Let us find isometries of the space of Finslerian 3-spinors, i.e., the linear transformations

$$\xi' = D\xi \quad \Longleftrightarrow \quad \xi'^a = d^a_b \xi^b \quad (D = \|d^a_b\|; d^a_b \in \mathbb{C}; a, b = 1, 2, 3) \tag{3}$$

which preserve the symplectic scalar 3-product:

$$[\xi', \eta', \lambda'] = [\xi, \eta, \lambda] \quad \text{for any} \quad \xi, \eta, \lambda \in \mathbb{C}^3. \tag{4}$$

Substituting (3) and the similar expressions for η', λ' into the condition (4), we obtain

$$[\xi, \eta, \lambda] \det D = [\xi, \eta, \lambda] \tag{5}$$

with regard to (2). Due to arbitrariness of ξ, η, $\lambda \in \mathbb{C}^3$, the equation (5) implies $\det D = 1$. Thus, the isometries of the space of Finslerian 3-spinors form the group $\mathrm{SL}(3, \mathbb{C})$.

Let us consider the subspace of the linear space $\mathbb{C}^3 \otimes \overline{\mathbb{C}^3}$ which consists of Hermitian tensors. This subspace is isomorphic to the 9-dimensional *real* linear space $\mathrm{Herm}(3) = \{X \mid X = X^+\}$ of all Hermitian 3×3 matrices with complex elements. Here and below, the over-line denotes complex conjugating, while the cross does Hermitian conjugating.

As a basis of the space Herm(3), we choose the following linearly independent matrices

$$\begin{array}{lll}
\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array} \tag{6}$$

($\lambda_1, \lambda_2, \ldots, \lambda_7$ are the well-known Gell-Mann matrices). Then, for any $X \in \mathrm{Herm}(3)$, we have the expansion

$$X = X^A \lambda_A \quad (A = 0, 1, \ldots, 8), \tag{7}$$

where $X^A \in \mathbb{R}$ are components of the 9-vector X with respect to the basis (6). Along with the matrices (6), we introduce another set of the Hermitian 3×3 matrices: $\lambda^B = \lambda_B$ $(B \neq 8)$, $\lambda^8 = 2\lambda_8$. Under such a choice of the matrices, the remarkable relations

$$\mathrm{Tr}(\lambda^A \lambda_B) = 2\delta^A_B \quad (A, B = 0, 1, \ldots, 8) \tag{8}$$

are fulfilled. Here, $\mathrm{Tr}(\lambda^A \lambda_B)$ denotes the trace of the matrix $\lambda^A \lambda_B$ and δ^A_B is the Kronecker symbol. Because of (7) and (8),

$$X^A = \frac{1}{2}\mathrm{Tr}(\lambda^A X). \tag{9}$$

Let us equip Herm(3) with the structure of the Finslerian space. To this end, we define the *length* $|X|$ of the 9-vector $X \in \mathrm{Herm}(3)$ in the following way:

$$|X| \equiv \sqrt[3]{\det X}.$$

Computing the determinant of (7), we obtain the expression for $|X|^3$ in the basis (6):

$$|X|^3 = G_{ABC}X^A X^B X^C = [(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2]X^8 - \\ - X^0[(X^4)^2 + (X^5)^2 + (X^6)^2 + (X^7)^2] + 2X^1[X^4X^6 + X^5X^7] + \\ + 2X^2[X^5X^6 - X^4X^7] + X^3[(X^4)^2 + (X^5)^2 - (X^6)^2 - (X^7)^2]. \quad (10)$$

Here, G_{ABC} are components of the covariant symmetric tensor on Herm(3). Thus, the Finslerian length of the 9-vector $X \in$ Herm(3) in the basis (6) is the form of degree 3 with respect to its components (9). It should be noted that the form (10) is indefinite, i.e., the cases $|X|^3 > 0$, $|X|^3 < 0$, or $|X|^3 = 0$ are possible. Since $|X|^3 = \det X$, we have $|X|^3 = 0$ if and only if $\det X = 0$.

Any linear transformation (3) of the space of Finslerian 3-spinors induces the transformation

$$X' = DXD^+ \quad \Longleftrightarrow \quad X'^{a\dot{b}} = d^a_c \overline{d^{\dot{b}}_{\dot{e}}} X^{c\dot{e}} \quad (X' = \|X'^{a\dot{b}}\|; X = \|X^{c\dot{e}}\|) \quad (11)$$

in Herm(3). Here, all the indices run from 1 to 3 and $X \in$ Herm(3). It is evident that the transformation (11) has the following properties:

1. If $X = X^+$, then $X' = X'^+$, i.e., $X \in$ Herm(3) implies $X' \in$ Herm(3).

2. The transformation (11) is linear with respect to X.

3. If $\det D = 1$, then $\det X' = \det X$ for any $X \in$ Herm(3).

Since $|X| = \sqrt[3]{\det X}$, the last property means that the linear transformation (11) with $D \in \mathrm{SL}(3, \mathbb{C})$ is a Finslerian isometry of the space Herm(3), i.e., $|X'| = |X|$. It is clear that all such isometries form a group. We will give the explicit matrix description of this group in the basis (6).

Let us substitute the expansions $X' = X'^A\lambda_A$ and $X = X^B\lambda_B$ into (11). We then multiply the resulting equality by λ^A from the left, compute its trace, and use the relations (8). As a result, we obtain

$$X'^A = L(D)^A_B X^B \quad (A, B = 0, 1, \ldots, 8), \quad (12)$$

where

$$L(D)^A_B = \frac{1}{2}\mathrm{Tr}(\lambda^A D \lambda_B D^+) \quad (13)$$

are elements of the matrix of the linear transformation (11) in the basis (6). It should be noted that $L(D)^A_B \in \mathbb{R}$. Thus, for any $D \in \mathrm{SL}(3, \mathbb{C})$, the transformation (12)–(13) preserves the form (10):

$$G_{ABC}X'^A X'^B X'^C = G_{ABC}X^A X^B X^C.$$

Since the group $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{SL}(3,\mathbb{C})$ is locally isomorphic to the group $\mathrm{O}_+^{\uparrow}(1,3)$ [7], it is interesting to consider the transformation (12)–(13) with $D \in \mathrm{SL}(2,\mathbb{C})$, i.e., from the point of view of a "4-dimensional observer". This will allow us to represent the expression (10) for the Finslerian length of the 9-vector completely in the 4-dimensional form.

Let

$$D_2 = \begin{pmatrix} d_1^1 & d_2^1 & 0 \\ d_1^2 & d_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det D_2 = 1 \quad (d_{\hat{b}}^{\hat{a}} \in \mathbb{C}; \hat{a}, \hat{b} = 1, 2). \tag{14}$$

The matrices (14) form a subgroup of $\mathrm{SL}(3,\mathbb{C})$ which is isomorphic to the group $\mathrm{SL}(2,\mathbb{C})$. Let us substitute the matrix D_2 from (14) into (13) instead of D. Direct computations show that

$$\begin{aligned}
L(D_2)_0^0 &= \frac{1}{2}(d_1^1\overline{d_1^1} + d_2^1\overline{d_2^1} + d_1^2\overline{d_1^2} + d_2^2\overline{d_2^2}),\\
L(D_2)_1^0 &= \frac{1}{2}(d_1^1\overline{d_2^1} + d_1^2\overline{d_2^2} + d_2^1\overline{d_1^1} + d_2^2\overline{d_1^2}),\\
L(D_2)_2^0 &= \frac{i}{2}(d_2^1\overline{d_1^1} + d_2^2\overline{d_1^2} - d_1^1\overline{d_2^1} - d_1^2\overline{d_2^2}),\\
L(D_2)_3^0 &= \frac{1}{2}(d_1^1\overline{d_1^1} + d_1^2\overline{d_1^2} - d_2^1\overline{d_2^1} - d_2^2\overline{d_2^2}),\\
L(D_2)_0^1 &= \frac{1}{2}(d_1^1\overline{d_1^2} + d_1^2\overline{d_1^1} + d_2^1\overline{d_2^2} + d_2^2\overline{d_2^1}),\\
L(D_2)_1^1 &= \frac{1}{2}(d_1^1\overline{d_2^2} + d_1^2\overline{d_2^1} + d_2^1\overline{d_1^2} + d_2^2\overline{d_1^1}),\\
L(D_2)_2^1 &= \frac{i}{2}(d_2^1\overline{d_1^2} + d_2^2\overline{d_1^1} - d_1^1\overline{d_2^2} - d_1^2\overline{d_2^1}),\\
L(D_2)_3^1 &= \frac{1}{2}(d_1^1\overline{d_1^2} + d_1^2\overline{d_1^1} - d_2^1\overline{d_2^2} - d_2^2\overline{d_2^1}),\\
L(D_2)_0^2 &= \frac{i}{2}(d_1^1\overline{d_1^2} - d_1^2\overline{d_1^1} + d_2^1\overline{d_2^2} - d_2^2\overline{d_2^1}),\\
L(D_2)_1^2 &= \frac{i}{2}(d_1^1\overline{d_2^2} - d_1^2\overline{d_2^1} + d_2^1\overline{d_1^2} - d_2^2\overline{d_1^1}),\\
L(D_2)_2^2 &= \frac{1}{2}(d_1^1\overline{d_2^2} + d_2^2\overline{d_1^1} - d_2^1\overline{d_1^2} - d_1^2\overline{d_2^1}),\\
L(D_2)_3^2 &= \frac{i}{2}(d_1^1\overline{d_1^2} - d_1^2\overline{d_1^1} - d_2^1\overline{d_2^2} + d_2^2\overline{d_2^1}),\\
L(D_2)_0^3 &= \frac{1}{2}(d_1^1\overline{d_1^1} - d_1^2\overline{d_1^2} + d_2^1\overline{d_2^1} - d_2^2\overline{d_2^2}),\\
L(D_2)_1^3 &= \frac{1}{2}(d_1^1\overline{d_2^1} - d_1^2\overline{d_2^2} + d_2^1\overline{d_1^1} - d_2^2\overline{d_1^2}),
\end{aligned}$$

$$L(D_2)^3_2 = \frac{i}{2}(d^1_2\overline{d^1_1} - d^2_2\overline{d^2_1} - d^1_1\overline{d^1_2} + d^2_1\overline{d^2_2}),$$
$$L(D_2)^3_3 = \frac{1}{2}(d^1_1\overline{d^1_1} - d^1_2\overline{d^1_2} - d^2_1\overline{d^2_1} + d^2_2\overline{d^2_2}), \qquad (15)$$

$L(D_2)^{3+i}_{3+j} = M(D_2)^i_j$ $(i, j = 1, 2, 3, 4)$, where

$$\begin{aligned}
M(D_2)^1_1 &= \frac{1}{2}(\overline{d^1_1} + d^1_1), & M(D_2)^3_1 &= \frac{1}{2}(\overline{d^2_1} + d^2_1),\\
M(D_2)^1_2 &= \frac{i}{2}(\overline{d^1_1} - d^1_1), & M(D_2)^3_2 &= \frac{i}{2}(\overline{d^2_1} - d^2_1),\\
M(D_2)^1_3 &= \frac{1}{2}(\overline{d^1_2} + d^1_2), & M(D_2)^3_3 &= \frac{1}{2}(\overline{d^2_2} + d^2_2),\\
M(D_2)^1_4 &= \frac{i}{2}(\overline{d^1_2} - d^1_2), & M(D_2)^3_4 &= \frac{i}{2}(\overline{d^2_2} - d^2_2),\\
M(D_2)^2_1 &= \frac{i}{2}(d^1_1 - \overline{d^1_1}), & M(D_2)^4_1 &= \frac{i}{2}(d^2_1 - \overline{d^2_1}),\\
M(D_2)^2_2 &= \frac{1}{2}(d^1_1 + \overline{d^1_1}), & M(D_2)^4_2 &= \frac{1}{2}(d^2_1 + \overline{d^2_1}),\\
M(D_2)^2_3 &= \frac{i}{2}(d^1_2 - \overline{d^1_2}), & M(D_2)^4_3 &= \frac{i}{2}(d^2_2 - \overline{d^2_2}),\\
M(D_2)^2_4 &= \frac{1}{2}(d^1_2 + \overline{d^1_2}), & M(D_2)^4_4 &= \frac{1}{2}(d^2_2 + \overline{d^2_2}),
\end{aligned} \qquad (16)$$

$L(D_2)^8_8 = 1$, while the other elements of the matrix of the transformation $X'^A = L(D_2)^A_B X^B$ vanish. Thus, for $D = D_2$, the Finslerian isometry (12) has the form

$$\begin{aligned}
X'^\alpha &= L(D_2)^\alpha_\beta X^\beta \quad (\alpha, \beta = 0, 1, 2, 3),\\
\theta'^i &= M(D_2)^i_j \theta^j \quad (i, j = 1, 2, 3, 4),\\
X'^8 &= X^8,
\end{aligned} \qquad (17)$$

where $L(D_2)^\alpha_\beta$, $M(D_2)^i_j$ are given by (15)–(16) and the notation $\theta'^i = X'^{3+i}$, $\theta^j = X^{3+j}$ is used.

It was shown in the paper [5] that (15) and (16) are the elements of the matrices of the transformations for a Lorentz 4-vector and a Majorana 4-spinor respectively. Therefore, the result (17) asserts that, for $D = D_2$, the 9-vector X^A splits into the Lorentz 4-vector X^α, the Majorana 4-spinor θ^i, and the Lorentz 4-scalar X^8.

This is the essence of the procedure of dimensional reduction allowing to display the "4-dimensional structure" of 9-dimensional expressions. Let us apply this procedure to the cumbersome formula (10) for the Finslerian

length of the 9-vector X^A. Taking into consideration (17), we obtain

$$|X|^3 = g_{\mu\nu}X^\mu X^\nu X^8 - g_{\mu\nu}X^\mu\overline{\theta}\gamma^\nu\theta, \tag{18}$$

where $\mu, \nu = 0, 1, 2, 3$, $\|g_{\mu\nu}\| = \mathrm{diag}\,(1, -1, -1, -1)$ is the matrix of components of the Minkowski metric tensor in a pseudoorthonormal basis,

$$\gamma^0 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu})$$

are the Dirac matrices in the Majorana representation [5], $\theta \in \mathbb{R}^4$ is the 4-component column of real numbers $\theta^j = X^{3+j}$ $(j = 1, 2, 3, 4)$, and $\overline{\theta} \equiv \theta^\top\gamma^0$ (the mark $\top$ denotes the matrix transposition). Thus, the expression (10) is written in the compact 4-dimensional form (18).

The generalized Duffin–Kemmer equation

Let us use the above formalism for the quantum description of a free 3-spinor particle in the 9-dimensional Finslerian space with the metric (10). The corresponding wave equation was obtained in the works [2, 8]. The paper [2] dealt with the coordinate representation, while the paper [8] did with the momentum representation of the same wave equation. Below, we use the momentum representation because the differential equations for wave functions of free particles become purely algebraic and, therefore, simpler to analyze in this representation.

Let i^r and $\beta_{\dot{s}}$ $(r, s = 1, 2, 3)$ be Finslerian 3-spinors, while $P \equiv \|P^{r\dot{s}}\|$ is an element of the space Herm(3). Substituting the matrix P instead of X into (7) and computing the determinant, we obtain

$$\det P = G_{ABC}P^A P^B P^C \tag{19}$$

in the notation of the formula (10). Here, the relations between $P^{r\dot{s}}$ and P^A have the form

$$\left.\begin{array}{lll} P^{1\dot{1}} = P^0 + P^3, & P^{1\dot{2}} = P^1 - iP^2, & P^{1\dot{3}} = P^4 - iP^5 \\ P^{2\dot{1}} = P^1 + iP^2, & P^{2\dot{2}} = P^0 - P^3, & P^{2\dot{3}} = P^6 - iP^7 \\ P^{3\dot{1}} = P^4 + iP^5, & P^{3\dot{2}} = P^6 + iP^7, & P^{3\dot{3}} = P^8 \end{array}\right\}. \tag{20}$$

In the work [8], to describe a free 3-spinor particle with the wave function $i^r(P^A)$, $\beta_{\dot{s}}(P^A)$, the SL$(3,\mathbb{C})$-covariant equation

$$\left.\begin{array}{r} P^{r\dot{s}}\beta_{\dot{s}} = Mi^r \\ P_{r\dot{s}}\, i^r = M^2\beta_{\dot{s}} \end{array}\right\} \tag{21}$$

was proposed. Here, $P^{r\dot{s}}$ are expressed by (20) in terms of the 9-momentum P^A of the particle, M is a positive scalar, and $P_{r\dot{s}}$ are the cofactors of the elements $P^{r\dot{s}}$ of the matrix P. It is natural to call M the 9-mass of the particle because substituting the upper equality of (21) into the lower one (and vice versa) gives a Finslerian analog of the Klein–Gordon equation for each 3-spinor component of the wave function:

$$(G_{ABC}P^AP^BP^C - M^3)i^r = 0, \quad (G_{ABC}P^AP^BP^C - M^3)\beta_{\dot{s}} = 0.$$

Assuming $P^{3+i} = 0$ $(i = 1,2,3,4)$, $P^8 = M$ and performing dimensional reduction as in (17), we conclude that the equation (21) splits into the standard 4-dimensional Dirac (for the components i^1, i^2, $\beta_{\dot{1}}$, $\beta_{\dot{2}}$) and Klein–Gordon (for the component $i^3 = \beta_{\dot{3}}$) equations in the momentum representation, which describe free particles with the mass M [8].

It should be noted that the equation (21) is quadratic with respect to P^A. This follows from (20) and the fact that $P_{r\dot{s}}$ are proportional to 2×2 minors of the matrix P. Let us try to represent (21) in the form of an equation, which is linear with respect to the 9-momentum P^A.

Following the work [9], we introduce the new variables ξ_1, ξ_2, …, ξ_6 such that

$$\left.\begin{array}{ll} P^{2\dot{1}}i^1 - P^{1\dot{1}}i^2 = M\xi_1, & P^{2\dot{2}}i^1 - P^{1\dot{2}}i^2 = M\xi_4 \\ P^{3\dot{1}}i^1 - P^{1\dot{1}}i^3 = M\xi_2, & P^{3\dot{2}}i^1 - P^{1\dot{2}}i^3 = M\xi_5 \\ P^{3\dot{1}}i^2 - P^{2\dot{1}}i^3 = M\xi_3, & P^{3\dot{2}}i^2 - P^{2\dot{2}}i^3 = M\xi_6 \end{array}\right\}. \tag{22}$$

With the help of (22), we can rewrite the lower equality of (21) in the form

$$\left.\begin{array}{r} P^{3\dot{3}}\xi_4 - P^{2\dot{3}}\xi_5 + P^{1\dot{3}}\xi_6 = M\beta_{\dot{1}} \\ -P^{3\dot{3}}\xi_1 + P^{2\dot{3}}\xi_2 - P^{1\dot{3}}\xi_3 = M\beta_{\dot{2}} \\ -P^{3\dot{1}}\xi_4 + P^{2\dot{1}}\xi_5 - P^{1\dot{1}}\xi_6 = M\beta_{\dot{3}} \end{array}\right\}. \tag{23}$$

Thus, (21) is equivalent to the set of equations $P^{r\dot{s}}\beta_{\dot{s}} = Mi^r$, (22)–(23) or, what is the same, to the matrix equation

$$\hat{P}\Psi = M\Psi, \tag{24}$$

where $\Psi = (i^1, i^2, i^3, \beta_{\dot{1}}, \beta_{\dot{2}}, \beta_{\dot{3}}, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)^\top$ is the 12-component column and

$$\hat{P} = \begin{pmatrix} \mathbf{0} & P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P_1 & P_2 \\ P_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ P_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{25}$$

is the 12×12 matrix consisting of the 3×3 blocks:

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

$$P = \begin{pmatrix} P^{1\dot{1}} & P^{1\dot{2}} & P^{1\dot{3}} \\ P^{2\dot{1}} & P^{2\dot{2}} & P^{2\dot{3}} \\ P^{3\dot{1}} & P^{3\dot{2}} & P^{3\dot{3}} \end{pmatrix}, \tag{27}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ -P^{3\dot{3}} & P^{2\dot{3}} & -P^{1\dot{3}} \\ 0 & 0 & 0 \end{pmatrix}, \tag{28}$$

$$P_2 = \begin{pmatrix} P^{3\dot{3}} & -P^{2\dot{3}} & P^{1\dot{3}} \\ 0 & 0 & 0 \\ -P^{3\dot{1}} & P^{2\dot{1}} & -P^{1\dot{1}} \end{pmatrix}, \tag{29}$$

$$P_3 = \begin{pmatrix} P^{2\dot{1}} & -P^{1\dot{1}} & 0 \\ P^{3\dot{1}} & 0 & -P^{1\dot{1}} \\ 0 & P^{3\dot{1}} & -P^{2\dot{1}} \end{pmatrix}, \tag{30}$$

$$P_4 = \begin{pmatrix} P^{2\dot{2}} & -P^{1\dot{2}} & 0 \\ P^{3\dot{2}} & 0 & -P^{1\dot{2}} \\ 0 & P^{3\dot{2}} & -P^{2\dot{2}} \end{pmatrix}. \tag{31}$$

Let us raise (25) to the fourth power. The direct calculation shows that

$$\hat{P}^4 = (\det P)\hat{P}. \tag{32}$$

On the other hand, by using (20) and (26)–(31), it is easy to represent (25) in the form of the linear combination

$$\hat{P} = P^A \delta_A \tag{33}$$

of the nine 12×12 matrices δ_A (in the explicit form, these matrices are given in the Appendix to the paper [9]). The substitution of (19) and (33) into (32) results in the identity

$$(P^A \delta_A)^4 = G_{ABC} P^A P^B P^C (P^D \delta_D) \tag{34}$$

valid for any P^A. Here, $A, B, C, D = 0, 1, \ldots, 8$.

It is evident that (34) generalizes the known 4-dimensional identity

$$(p^\mu \beta_\mu)^3 = g_{\mu\nu} p^\mu p^\nu (p^\lambda \beta_\lambda),$$

where $\mu, \nu, \lambda = 0, 1, 2, 3$ and β_μ are the Duffin–Kemmer matrices [10, 11]. Moreover, it follows from (34) that the δ-matrices satisfy the conditions

$$\delta_{(A}\delta_B\delta_C\delta_{D)} = 6\{G_{ABC}\delta_D + G_{ABD}\delta_C + G_{ACD}\delta_B + G_{BCD}\delta_A\}, \tag{35}$$

where the parentheses denote the symmetrization with respect to all the subscripts (i.e., the sum over all permutations of A, B, C, D). In this connection, it is interesting to recall important relations of the Duffin–Kemmer algebra:

$$\beta_{(\mu}\beta_\nu\beta_{\lambda)} = 2\{g_{\mu\nu}\beta_\lambda + g_{\lambda\mu}\beta_\nu + g_{\lambda\nu}\beta_\mu\}. \tag{36}$$

It is easy to see the full analogy between the formulas (35) and (36).

Let us return to the equation (24). With the help of (33), it can finally be written in the following form

$$(P^A\delta_A - M)\Psi = 0, \tag{37}$$

where δ_A satisfy the conditions (35). Thus, the purpose of this section is achieved: the equation (21) is represented in the form of the generalized Duffin–Kemmer equation (37).

Conclusion

Summarizing, we make some remarks concerning the obtained results.

In this paper, the main facts of the geometry of Finslerian 3-spinors of the 9-dimensional linear space with the metric function defined by the cubic form (10) are formulated. The explicit description of isometries of this 9-dimensional Finslerian space and the procedure of dimensional reduction, which allowed us to represent (10) in the 4-dimensional form (18), are given. The latter is important because it demonstrates the correspondence of our constructions to the standard relativistic theory on the level of geometry.

In addition, we deduced the generalized Duffin–Kemmer equation (37) for a free Finslerian 3-spinor particle in the momentum representation. In a parallel way, we obtained the 9-dimensional Finslerian analog (35) of the defining relations (36) of the 4-dimensional Duffin–Kemmer algebra. It is also shown that the equation (37) unifies the 4-dimensional Dirac and Klein–Gordon equations in a nontrivial way.

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